A large deviation principle for a RWRC in a box

IRTG seminar

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1. Large Deviations for dummies
   - Cramér's Theorem
   - Main definition
   - Applications
2. Random Walk among Random Conductances
   - The model
   - Related fields
   - The main theorem
   - Sketch of the proof
3. Future work

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- Large Deviation Principle (LDP) (+ other conditions):

$$\mathbb{P}\Big(\frac{1}{n}\sum_{j=1}^{n} X_j \geq x\Big) \approx \mathrm{e}^{-n\mathcal{I}(x)}.$$

Large Deviation Theory deals with asymptotic computation of small probabilities on an exponential scale.

Theorem (Cramér)

Let $X_1, X_2, \ldots$ be i.i.d. $\mathbb{R}$-valued random variables such that

$$\varphi(t) = \mathbb{E}\left[e^{tX_1}\right] < \infty \qquad \forall t \in \mathbb{R}.$$

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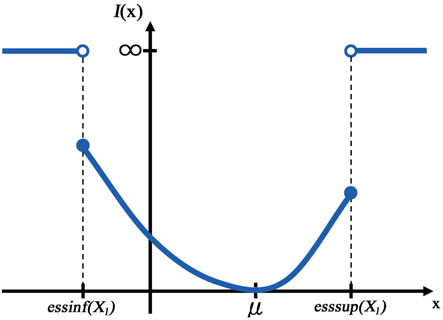
Then, for all $x > \mathbb{E}[X_1]$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{1}{n}\sum_{j=1}^{n} X_j \geq x\Big) = -\mathcal{I}(x),$$

where

$$\mathcal{I}(x) := \sup_{t\in\mathbb{R}} \left[tx - \log \varphi(t)\right].$$


I(x)
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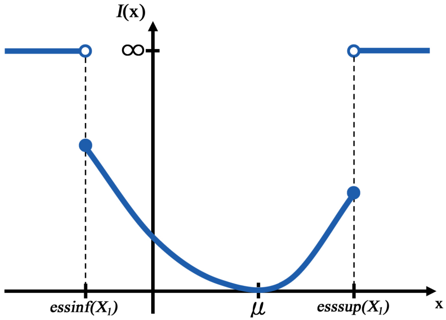

The function $\mathcal{I}(x)$

- is convex,
- has compact level sets ($\Longrightarrow$ is lower semi-continuous),
- $\mathcal{I}(x) \geq 0$ and equality holds iff $x = \mu = \mathbb{E}[X_1]$.

Definition

Let $\mathcal{X}$ be a Polish space. A function $\mathcal{I} : \mathcal{X} \to [0, \infty]$ is called rate function if

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1. *For every open set O,* $$\liminf_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(O) \geq - \inf_{x\in O} \mathcal{I}(x);$$
2. *For every closed set C,* $$\limsup_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(C) \leq - \inf_{x\in C} \mathcal{I}(x).$$

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5. in general no relation between LDP and CLT.

Recall "Laplace method":

$$\forall f : [0,1] \to \mathbb{R} \text{ continuous}, \qquad \lim_{n\to\infty} \frac{1}{n} \log \int_0^1 \mathrm{e}^{nf(x)} \mathrm{d}x = \max_{x\in[0,1]} f(x).$$

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Theorem (Varadhan's Lemma)

Let (μ_n) satisfy an LDP on the Polish space $\mathcal{X}$ with speed γ_n and rate function $\mathcal{I}$. Let $F : \mathcal{X} \to \mathbb{R}$ be a continuous function bounded from above. Then

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \log \int_{\mathcal{X}} \mathrm{e}^{\gamma_n F(x)} \mu_n(\mathrm{d}x) = \sup_{x\in\mathcal{X}} \left[F(x) - \mathcal{I}(x) \right].$$

Let $(X_t)_{t\in[0,\infty)}$ be the simple random walk on $\mathbb{Z}^d$ in continuous time. Its generator is the Laplace operator:

$$\Delta f(x) = \sum_{y\in\mathbb{Z}^d:\, y\sim x} [f(y) - f(x)], \quad x \in \mathbb{Z}^d,\, f : \mathbb{Z}^d \to \mathbb{R}$$

Call empirical measure

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Theorem (Donsker, Varadhan)

The process of empirical measures $(\frac{1}{t}\ell_t)_{t\in\mathbb{R}^+}$ of the simple random walk under $\mathbb{P}_0(\,\cdot\, \cap \{\mathrm{supp}(\ell_t) \subseteq B\})$ satisfies a large deviation principle on $\mathcal{M}_1(B)$ with speed $\gamma_t = t$ and rate function

$$I(\mu) = \langle -\Delta_B \sqrt{\mu}, \sqrt{\mu}\rangle = \frac{1}{2} \sum_{x\sim y:\, x\in B} \left(\sqrt{\mu(x)} - \sqrt{\mu(y)}\right)^2.$$

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- $\omega_{x,e} = \omega_{x+e,-e}$ (symmetry),
- $\{\omega_{x,e}\}_{x\in\mathbb{Z}^d, e\in\mathcal{E}}$ are i.i.d.,
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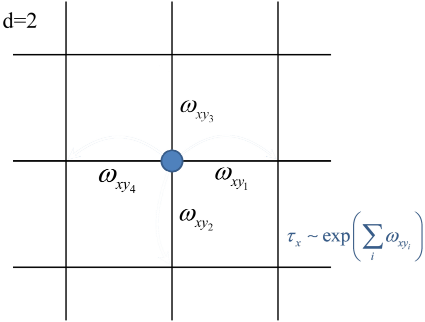

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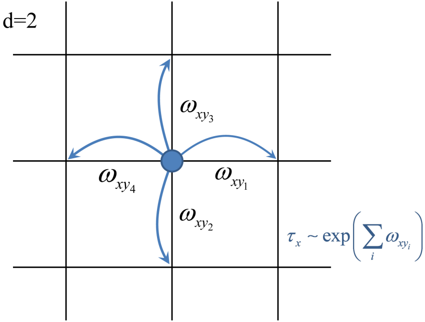

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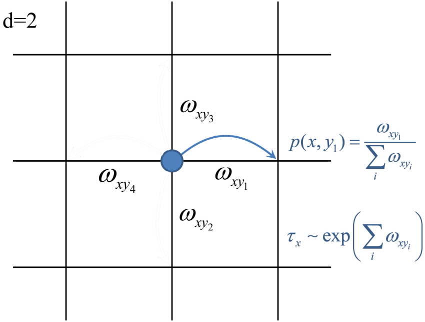

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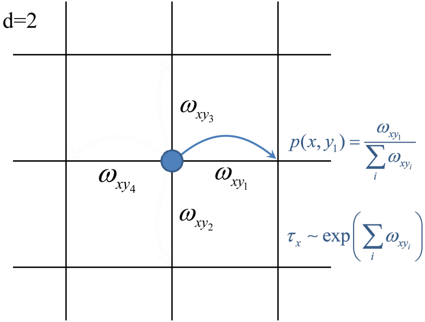

Definition

The Random Walk among Random Conductances (RWRC) is the continuous-time process generated by

$$\Delta^\omega f(x) := \sum_{x\in\mathbb{Z}^d, e\in\mathcal{E}} \omega_{x,e}\big(f(x+e) - f(x)\big).$$

	RWRE	RWRC
Time	Mostly discrete	Mostly continuous
Reversibility	No	Yes
Problems	CLT, SLLN, criteria for transience/recurrence, ballisticity	CLT, SLLN

What has been done so far?

- $\omega_{x,e} \in [0,1]$ with $\mathrm{Prob}(\omega_{x,e} > 0) > p_c(d)$, discrete time: quenched functional CLT, via homogenization [BISKUP, PRESCOTT (2007)]

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- Aim: large deviation principle for local times!

Let $(X_t)_{t\in[0,\infty)}$ be the RWRC. For $x \in B \subseteq \mathbb{Z}^d$, B finite and connected set, define the local time

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We want to study the annealed behaviour of ℓ_t:

$$\left\langle \mathbb{P}_0^\omega\left(\frac{1}{t}\ell_t \sim g^2\right)\right\rangle$$

where $g : B \to \mathbb{R}^+$, with $\operatorname{supp}(g) \subseteq B, \sum_{x\in B} g^2(x) = 1$ and $\langle \cdot \rangle$ is the expectation w.r.t. the conductances.

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Three noises:

- ★ the conductances;
- ★ the waiting times;
- ★ the embedded discrete-time RW.

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- Note that, by a Fourier expansion:

$$\mathbb{P}_0^\omega\Big(\frac{1}{t}\ell_t \sim g^2 \,\Big|\, \mathrm{supp}(\ell_t) \in B\Big) = \sum_{k=1}^{d\cdot\#B} \mathrm{e}^{t\lambda_k^\omega(B)} f_k(0)\langle f_k, \mathbb{1}\rangle \approx \mathrm{e}^{t\lambda_1^\omega(B)},$$

where $\lambda_1^\omega(B)$ is the bottom of the spectrum of $-\Delta^\omega$ restricted to the box B. Relation with Random Schrödinger operators!

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- Parabolic Anderson model with random Laplace operator:

$$\begin{cases} \partial_t u(x,t) & = \Delta^\omega u(x,t) + \xi(x)u(x,t), \quad t \in (0,\infty),\ x \in \mathbb{Z}^d \\ u(x,0) & = \delta_0(x) \qquad\qquad\qquad\qquad\qquad\quad x \in \mathbb{Z}^d. \end{cases}$$

Feynman-Kac formula gives $u(x,t) = \mathbb{E}_x^\omega\Big[\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s}\delta_0(X_t)\Big]$, where X_t is a RWRC.

Recall:

- $B \subseteq \mathbb{Z}^d$ finite and connected;
- $\log \mathrm{Pr}\big(\omega_{x,e} < \varepsilon\big) \approx -\varepsilon^{-\eta}, \quad$ for $\varepsilon \downarrow 0$, $\eta > 1$;
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Theorem (joint work with Wolfgang König and Tilman Wolff)

The process of empirical measures $(\frac{1}{t}\ell_t)_{t\in\mathbb{R}^+}$ *of the Random Walk among Random Conductances under the annealed law* $\big\langle \mathbb{P}_0^\omega(\,\cdot\, \cap \{\mathrm{supp}(\ell_t) \subseteq B\})\big\rangle$ *satisfies a large deviation principle on* $\mathcal{M}_1(B)$ *with speed* $\gamma_t = t^{\frac{\eta}{\eta+1}}$ *and rate function* J *given by*

$$J(g^2) := C_\eta \sum_{z,e} |g(z+e) - g(z)|^{\frac{2\eta}{\eta+1}} = C_\eta \|\nabla g\|_{\frac{2\eta}{1+\eta}}^{\frac{2\eta}{1+\eta}}$$

for all $g^2 \in \mathcal{M}_1(B)$, *where* $C_\eta := \big(1 + \frac{1}{\eta}\big)\eta^{\frac{1}{1+\eta}}$.

This means

$$\left\langle \mathbb{P}_0^\omega\Big(\Big\{ \frac{1}{t}\ell_t \sim g^2 \Big\} \cap \big\{ \mathrm{supp}(\ell_t) \subseteq B \big\} \Big) \right\rangle \approx \mathrm{e}^{-\gamma_t C_\eta \sum_{z,e} |g(z+e)-g(z)|^{\frac{2\eta}{\eta+1}}}.$$

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In particular:

Corollary

The annealed probability of non-exit from the box B for the Random Walk among Random Conductances for $t \gg 0$ is

$$\log \left\langle \mathbb{P}_0^\omega \Big(\mathrm{supp}(\ell_t) \subseteq B \Big) \right\rangle \simeq \sup_{g^2 \in \mathcal{M}_1(B)} -t^{\frac{\eta}{1+\eta}} C_\eta \sum_{z,e} |g(z+e) - g(z)|^{\frac{2\eta}{\eta+1}}.$$

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- optimization over the rescaled shape of the conductances.

$$\left\langle \mathbb{P}_0^{\omega}\Big(\frac{1}{t}\ell_t \sim g^2\Big)\mathbb{1}_{\{t^r\omega\sim\varphi\}}\right\rangle \approx \mathbb{P}_0^{t^{-r}\varphi}\Big(\frac{1}{t}\ell_t \sim g^2\Big)\mathrm{Prob}\big(t^r\omega \sim \varphi\big)$$

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LDP for the conductances:

$$\mathrm{Prob}\big(\forall z,e : t^r\omega_{z,e} \sim \varphi(z,e)\big) = \prod_{z,e}\mathrm{Prob}\big(\omega_{z,e} \sim t^{-r}\varphi(z,e)\big)$$
$$\approx \exp\Big\{ - t^{r\eta}\sum_{z,e}\varphi(z,e)^{-\eta}\Big\}.$$

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LDP for weighted random walk:

$$\mathbb{P}_0^\psi\Big(\frac{1}{t}\ell_t \sim g^2\Big) \approx \exp\Big\{-t\sum_{z,e}\psi(z,e)\big(g(z+e)-g(z)\big)^2\Big\}$$

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therefore

$$\mathbb{P}_0^{t^{-r}\varphi}\Big(\frac{1}{t}\ell_t \sim g^2\Big) = \mathbb{P}_0^\varphi\Big(\frac{1}{t^{1-r}}\ell_{t^{1-r}} \sim g^2\Big)$$
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Physicists' trick:

Best rate of convergence for

$$t^{r\eta} \asymp t^{1-r}, \text{ i.e. } r = \frac{1}{1+\eta}.$$

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Optimization over φ for fixed g:

$$\sum_{z,e} \left[\varphi(z,e)^{-\eta} - \varphi(z,e)(g(z+e)-g(z))^2\right]$$

is optimal if

$$\varphi(z,e) = \eta^{\frac{1}{1+\eta}} |g(z+e)-g(e)|^{-\frac{2}{1+\eta}}.$$

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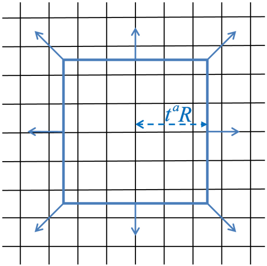
- Upper bound for closed sets.
  Problem: $t^r\omega$ is not bounded. We need a compactification argument for the space of rescaled conductances.

Future work:



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$$B_t = \alpha_t \, B \cap \mathbb{Z}^d,$$

say $\alpha_t = t^a$.

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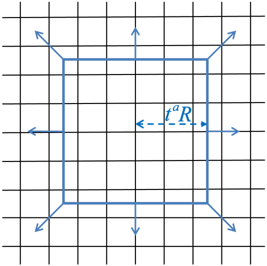

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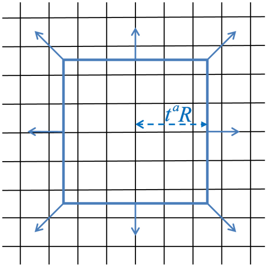

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L_t should satisfy an LDP with speed

$$\gamma_t = t^{\frac{1-a(d\eta-2)}{1+\eta}}$$

and rate function

$$J(f^2) = C_\eta \sum_{e \in \mathcal{N}_+} \int_B \left((e \cdot \nabla f)^2 \right)^{\frac{\eta}{1+\eta}} = C_\eta \|\nabla f\|_{\frac{2\eta}{1+\eta}}^{\frac{2\eta}{1+\eta}}.$$

Dankeschön!